MTH 166

Lecture-1

Exact Differential Equations (EDE)

Unit 1: Ordinary Differential Equations

(**Book:** Higher Engineering Mathematics by Dr. B.S. Grewal,**Chapter-11**)

Topic:

Exact Differential Equations (EDE)

Learning Outcomes:

1. What is the form of Exact Differential Equations.

2. What is the Necessary and Sufficient condition for an equation to be called as Exact differential equations.

3. How to solve an Exact differential equation.

Exact Differential Equation (EDE):

An equation of the form:

$$M(x,y)dx + N(x,y)dy = 0 \qquad (1)$$

is called an exact differential equation if and only if:

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (Necessary and Sufficient Condition for EDE)

The solution of equation (1) is given by:

$\int_{y=con.} M(x,y)dx + \int (Terms\ of\ N\ free\ from\ x)dy = C$(Constant)

<u>**Exact Differential Equation (EDE):**</u>

EDE Form:

$$Mdx + Ndy = 0$$

Necessary and Sufficient Condition:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solution of EDE:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

Solve the following differential equations:

Problem 1: $\boldsymbol{(x^2 - ay)dx = (ax - y^2)dy}$

Solution: $(x^2 - ay)dx + (y^2 - ax)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (x^2 - ay) \Rightarrow \frac{\partial M}{\partial y} = 0 - a(1) = -a$

And $N = (y^2 - ax) \Rightarrow \frac{\partial N}{\partial x} = 0 - a(1) = -a$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation.

Hence solution of equation (1) is given by:

$$\int_{y=con.} M dx + \int (Terms\ of\ N\ free\ from\ x) dy = C$$

$$\Rightarrow \int_{y=con.} (x^2 - ay) dx + \int (y^2) dy = C$$

$$\Rightarrow \frac{x^3}{3} - ay(x) + \frac{y^3}{3} = c$$

$$\Rightarrow x^3 - 3axy + y^3 = 3c$$

$\Rightarrow x^3 - 3axy + y^3 = c_1$ **Answer.** $(3c = c_1)$

Problem 2: $\boldsymbol{\left(x^2-4xy-2y^2\right)dx+\left(y^2-4xy-2x^2\right)dy=0}$

Solution: $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0 \quad (1)$

Compare it with: $Mdx+Ndy=0$

Here $M=(x^2-4xy-2y^2) \Rightarrow \frac{\partial M}{\partial y}=-4x-4y$

And $N=(y^2-4xy-2x^2) \Rightarrow \frac{\partial N}{\partial x}=-4y-4x$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation

Hence solution of equation (1) is given by:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

$$\Rightarrow \int_{y=con.} (x^2 - 4xy - 2y^2)dx + \int (y^2)dy = C$$

$$\Rightarrow \frac{x^3}{3} - 4y\left(\frac{x^2}{2}\right) - 2y^2(x) + \frac{y^3}{3} = c$$

$$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = 6c$$

$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = c_1$ **Answer.** $(6c = c_1)$

Problem 3: $(\boldsymbol{y}\sin 2\boldsymbol{x})\boldsymbol{dx} - \left(\mathbf{1} + \boldsymbol{y}^{2} + \boldsymbol{cos}^{2}\boldsymbol{x}\right)\boldsymbol{dy} = \mathbf{0}$

Solution: $(y\sin 2x)dx - (1 + y^{2} + cos^{2}x)dy = 0 \qquad (1)$

Compare it with: $Mdx + Ndy = 0$

Here $M = (y\sin 2x) \Rightarrow \frac{\partial M}{\partial y} = \sin 2x$

And $N = -(1 + y^{2} + cos^{2}x) \Rightarrow \frac{\partial N}{\partial x} = -(2\cos x(-\sin x)) = \sin 2x$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation

Hence solution of equation (1) is given by:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

$$\Rightarrow \int_{y=con.} (y \sin 2x)dx - \int (1 + y^2)dy = C$$

$$\Rightarrow y\left(\frac{-\cos 2x}{2}\right) - y - \frac{y^3}{3} = c$$

$$\Rightarrow -3ycos2x - 6y - 2y^3 = 6c$$

$\Rightarrow 3ycos2x + 6y + 2y^3 + c_1 = 0$ **Answer.** $(6c = c_1)$

Problem 4: $\boldsymbol{(sec\,x\,tan\,x\,tan\,y - e^{x})dx + \left(sec\,x\,sec^{2}y\right)dy = 0}$

Solution: $(sec\,x\,tan\,x\,tan\,y - e^{x})dx - (sec\,x\,sec^{2}y)dy = 0$ (1)

Compare it with: $Mdx + Ndy = 0$

Here $M = (sec\,x\,tan\,x\,tan\,y - e^{x}) \Rightarrow \frac{\partial M}{\partial y} = secx\,tanx\,(sec^{2}y)$

And $N = (sec\,x\,sec^{2}y) \Rightarrow \frac{\partial N}{\partial x} = (secx\,tanx)sec^{2}y$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, so equation (1) is an exact differential equation

Hence solution of equation (1) is given by:

$$\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$$

$$\Rightarrow \int_{y=con.} (secx\ tanx\ tany\ - e^{x})dx + \int (0)dy = C$$

$\Rightarrow tany(secx) - e^{x} = c$ **Answer.**

Some examples from your text book

Example 11.27. *Solve* $(1 + 2xy \cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0.$

Solution. Here $M = 1 + 2xy \cos x^2 - 2xy$ and $N = \sin x^2 - x^2$

$$\therefore \qquad \frac{\partial M}{\partial y} = 2x \cos x^2 - 2x = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{const})} M dx + \int (\text{terms of } N \text{ not containing } x) = c$$

i.e.,
$$\int_{(y\ \text{const})} (1 + 2xy \cos x^2 - 2xy)\, dx = c \quad \text{or} \quad x + y\left[\int \cos x^2 \,.\, 2x dx - \int 2x\, dx\right] = c$$

or
$$x + y \sin x^2 - yx^2 = c.$$

Example 11.28. *Solve* $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$ *(Kurukshetra, 2005)*

Solution. Given equation can be written as

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0.$$

Here $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x.$

$$\therefore \qquad \frac{\partial M}{\partial y} = \cos x + \cos y + 1 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{const.})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e.,

$$\int_{(y\ \text{const.})} (y\cos x + \sin y + y)\,dx + \int (0)\,dx = c \quad \text{or} \quad y\sin x + (\sin y + y)\,x = c.$$

Example 11.25. *Solve* $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0.$ *(V.T.U., 2006)*

Solution. Here $M = y^2 e^{xy^2} + 4x^3$ and $N = 2xy\,e^{xy^2} - 3y^2$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y\,e^{xy^2} + y^2\,e^{xy^2}\,.\,2xy = \frac{\partial N}{\partial x}$$

Thus the equation is exact and its solution is

$$\int_{(y\ \text{const.})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

i.e.,

$$\int_{(y\ \text{const.})} (y^2 e^{y^2 x} + 4x^3)\,dx + \int (-3y^2)\,dy = c \quad \text{or} \quad e^{xy^2} + x^4 - y^3 = c.$$

Example 11.26. *Solve* $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\} dx + (x + \log x - x \sin y)\,dy = 0.$

(Marathwada, 2008 S ; V.T.U., 2006)

Solution. Here $M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$$\therefore \quad \frac{\partial M}{\partial y} = 1 + 1/x - \sin y = \frac{\partial N}{\partial x}$$

Then the equation is exact and its solution is

$$\int_{(y\ \text{const})} M dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

$$\int_{(y\ \text{const})} \left\{\left(1 + \frac{1}{x}\right) y + \cos y\right\} dx = c \quad \text{or} \quad (x + \log x)\,y + x \cos y = c.$$